Inductive Reasoning, Miracles, and Examples from Number Theory

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Introduction

It is an obvious fact that any empirical knowledge of physical laws currently possessed by human civilization arose, not from constant observation of the universe at all points in space at all instances in time, but rather through experiments conducted at specific points in space at specific instances in time. In fact, given the relatively small proportion of human beings who are engaged in science research, and given that even scientists do not spend every moment of their time performing experiments, it should be obvious to anyone that human beings are not constantly checking the laws of physics to prove rigorously that exceptions to known knowledge are not occurring. If one additionally considers the relatively short span of time modern human civilization has existed relative to the age of the earth, one realizes the great lengths of time that have passed with human beings not observing the universe. Nevertheless, the laws of physics are assumed to hold at all points in space and time. What then justifies this assumption?

For one thing, we can claim that it is simply an aspect of human psychology that, after constant exposure to a statistical regularity, we are inclined to apply inductive reasoning and expect that such a regularity must necessarily apply as universally as possible. In this paper I wish to augment that fact by pointing out that the relatively short lifespan of a human being relative to the age of the universe is somewhat analogous to situations in number theory. It is frequently the case that arbitrarily long strings of consecutive integers satisfy a given property, and yet, infinitely many positive integers also fail to satisfy that same property. I intend to demonstrate that by analogy it is only subjective human experience that causes people to feel that since they have not seen a miracle, that, therefore, miracles do not occur. In fact, comparing the situation in number theory with the situation of a human being observing the universe, the analogy is particularly strong, because a person who believes in the possibility of miracles need in fact only believe in a finite number of them having ever occurred, while in the examples from number theory, there are possibly infinitely many positive integers not having the given property. Furthermore the modern human lifespan is relatively constant, whereas in number theory one can give arbitrarily long strings of positive numbers having a given property (analogously to having a human being live arbitrarily long in an attempt to observe the laws of physics in the far future).

Philosophical preliminaries

Since I am more concerned in this paper with the frequency of the occurrences of miracles than with their interpretation, I can limit myself to only one aspect of the subject and define a miracle as an exception to the laws of physics. On the other hand, I shall define naturalism as the belief that no miracles can occur, and I am defending a position opposed to the case made for naturalism. My philosophical assumption is that miracles do occur, but that they are relatively rare. If one thinks of space-time as a manifold, then the subset of the manifold where miracles occur must be relatively small. Therefore it is not surprising that miracles are not often observed. My exposition will make it clear that no decision between our position and that of naturalism can possibly be made by conducting physics experiments.

Examples from number theory

In mathematics, the term *number theory* refers to a particular area of math. Number theory is in fact replete with examples where the following occurs: a property is defined (viz. a specific function is applied), and a number can either satisfy the property or not. Then it is frequently the case that arbitrarily long strings of integers exist which satisfy the given property, and yet infinitely many integers do not satisfy the given property. How is this phenomenon analogous to the situation with miracles? A human lifespan is so short compared to longer time periods, such as human history, the age of the earth or the age of the universe, that any statistical regularity observed during one's lifetime might not at all constitute a statistical regularity based on data taken from longer time periods. In other words, what appears to be a statistical regularity does so only because the data are taken from a very small sample size, and the small sample size is precisely due to the limitations placed on the knowledge of the observer due to human constraints.

To make the analogy with number theory more concrete, let us define the property to be that of being composite. A composite number is one that can be factorized into two parts; for example, it can be thought of as $\mathbf{n} = \mathbf{ab}$ where neither a nor b is equal to 1. To confirm that an arbitrarily long string of composite numbers can exist, we use the mathematical object called a factorial, the product of all positive integers less than or equal to a given number, designated as $\mathbf{n!}$. Thus, for example, $\mathbf{5! = 1 \times 2 \times 3 \times 4 \times 5 = 120.}$

Now consider the series of numbers $\mathbf{n!+2, n!+3, n!+4, ...n!+n.}$ There are $\mathbf{n-1}$ consecutive numbers in this string, and they are all composite. Simply pick a large number to represent **n**, and suppose that an observer is looking at such a string for that given **n** and trying to see if any numbers that are generated are prime numbers (integers that are divisible without remainders only by themselves and 1). It is the case that there are infinitely many primes, and yet he would not find any simply by looking at the numbers consecutively if **n** is large enough. So what appears to be a statistical regularity (long strings of composite numbers) is in fact not at all a statistical regularity when the entire set of natural numbers is considered.

Alternatively, one need not look at the entire set of natural numbers. One can look at an infinite subset of the natural numbers, and find a similar phenomenon occurring: a property exists with a long string of numbers having that property, and yet there are infinitely many numbers not having that property. An example is primes in an arithmetic progression. Recently, Ben Green and Terrence Tao proved that the prime numbers contain arbitrarily long arithmetic progressions. [1] Simply pick any number **k.** Then one can find an arithmetic progression, viz. **an + b** with **k** consecutive members of that arithmetic progression all of which are prime. Yet this arithmetic progression also has infinitely many composite numbers. [2]

On the possibility of various tests for various phenomena

Clearly, the above examples show it is fallacious reasoning to make generalizations based on a small sample size. In the case of the number theoretic examples, it is possible to produce instances where any arbitrarily large sample size is in fact too small. For a person who observes the universe, the sample size of observations he can possibly make is extremely small relative to the age of the universe. At the very least, intellectual honesty demands that the possibility of a miracle be admitted in theory. But a belief in the possibility of miracles raises new problems. A belief in naturalism would imply that the laws of physics at all points in space-time can be inferred by conducting a few experiments in the modern day. A belief in the possibility of miracles would mean that statistical irregularities can arise in the laws of physics, and if these irregularities arise outside the local neighborhood of space-time of an observer, how is that observer going to know when such an irregularity occurs? One possibility is that such an irregularity occurred in the past, and that such irregularities were recorded in texts which survived to the present day. We claim that the issue is then no longer an abstract philosophical one but involves actual examination of the specific text. We also claim that to simply dismiss all such texts is not a valid solution for the simple reason that since by definition such irregularities are not replicable, a dismissal of all such texts almost necessitates an inability to tell whether such an irregularity occurred.

What we mean is the following: For replicable phenomena such as the laws of physics, the matter can be decided simply by conducting an experiment. But for non-replicable irregularities in the laws of physics, no such test can be conducted. While one might argue that a physics experiment is a more reliable test than the examination of an ancient text, the very nature of a non-replicable irregularity means that a physics experiment is not possible, and hence one must look at the next best alternative. It is simply a case of different tests existing for the judgment of various phenomena, with these tests being of differing levels of reliability. The alternatives one can choose are 1) believing only phenomena which can be confirmed by the most reliable tests and not to believe anything else, or 2) choosing the most reliable tests that can possibly be applied for a given phenomenon. Clearly the second alternative yields a finer algorithm and it agrees with the first alternative on all reproducible phenomena.

Comparing miracles with the continuum hypothesis

A relevant topic to discuss at this point that illustrates the problem when the methods at one's disposal are insufficient to test a given phenomenon is the *continuum hypothesis*. The continuum hypothesis is basically a question about sizes of infinite sets, arising out of the work of Georg Cantor who showed that there are in some sense more real numbers than integers. Although it is obvious that there are both infinitely many integers and infinitely many real numbers, what Cantor showed was that if one were to assign an integer to each real number between 0 and 1, then no matter how one made the assignment, there was always a way to construct a real number that did not have an integer assigned to it. The question naturally arose, if infinite sets could have different sizes, whether there was an infinite set in some sense larger than the integers but smaller than the real numbers. In this case, standard methods of working with sets were insufficient. These standard methods comprise *Zermelo-Fraenkel* set theory along with the *Axiom of Choice,* which allows one to choose one element from each set from an infinite collection of sets. The work of Kurt Gödel and Paul Cohen showed that the continuum hypothesis is actually independent of Zermelo-Fraenkel set theory along with the Axiom of choice. [3],[4] That means that, if the continuum hypothesis is really a question that can be resolved with a sufficiently good set of axioms, then restricting ones tools to the axioms of Zermelo-Fraenkel set theory, along with the Axiom of choice will not be good enough to give an answer. Before proceeding, one should note, of course, that the question of whether miracles occur or not is one that actually has an answer. This author is not sure whether the continuum hypothesis really has an answer. Let us assume that it does. Then the similarity between this situation and the one with miracles becomes clear. With miracles, it is simply the case that conducting experiments in the laboratory cannot give an answer concerning whether or not miracles can occur. More powerful methods are needed. With the continuum hypothesis, if it really has an answer, then more axioms are needed to resolve the question, beyond the axioms of Zermelo-Fraenkel set theory and the axiom of choice. But with respect to miracles, what are these more powerful methods needed to actually resolve the question? One method as mentioned earlier is the actual examination of ancient texts which record the performance of miracles in history. Another possibility is the issue of faith. A famous quotation from philosophy, *Credo ut intellegam* may have relevance to this. The point here is that a belief in the worldview provided by the Bible makes it feasible for miracles to occur. [5] Of course, once a miracle is actually experienced, the individual already has direct proof of their possibility. One way to think of the situation is that a person who believes what is written in the Bible (in addition to believing that some knowledge can be gained from the conduct of science experiments) is equipped with a larger set of axioms. Using these additional axioms he holds a lot of beliefs which cannot be obtained from the traditional conduct of science. However the act of living with such a belief in the Bible creates the possibility for statistical irregularities to occur in his local neighborhood of space-time (not necessarily violations of physical law although those are certainly possible) which then additionally confirm that the worldview provided by the Bible is correct.

Records of Miracles in Ancient Texts

How can one think of records of miracles in the Bible in a way that highlights the similarity with mathematics? The point is that the historical events in the Bible occurred outside the local neighborhood of space-time of any observer living today. Now imagine a long string of consecutive composite numbers, so long that going through them in order would take longer than the lifespan of a modern human being. Nevertheless, one knows, simply from Euclid's proof of the infinitude of primes, that there are primes larger than the composite numbers in his string, even if he cannot actually produce any concretely. Likewise, even in the absence of a miracle in the local neighborhood of an observer, historical records exist in the Bible which testify that miracles have occurred in history.

An Obsession with Materialism

We are going to conclude by making a suggestion concerning attitudes towards miracles. We suggest that from a psychological perspective, one of the issues with miracles is that we human beings have no control over them. In the modern world, we are accustomed to various phenomena functioning deterministically. We know that the light will go on if the light switch is turned on, we know that in operating various kinds of mechanical equipment, the machine functions in a deterministic way, and we know how to modify computer programs to get the kind of result that we want. We do not mean to suggest that there is anything wrong with determinism; we only want to suggest that there is a huge difference between saying that it is hard to motivate oneself to think about phenomena which one cannot control and saying that phenomena which one cannot control do not exist.

Essentially, one of the main criticisms of the intelligent design movement was that the theory did not generate anything concrete scientifically in terms of applications. While it is always good to have applicability, it remains true that questions of existence are somewhat different and clearly cannot be reduced to questions of applicability in the material world. In mathematics, an interesting phenomenon is that one often knows about the existence of complicated mathematical objects without being able to say too much about them concretely. A classic example happened when David Hilbert used an existence proof in proving his basis theorem [6]. The proof initially met with resistance because it was not a constructive proof. Nevertheless, the mathematician Felix Klein recognized the importance and validity of such a proof and it later gained general recognition. In mathematics, accepting the Law of the Excluded Middle and the Axiom of Choice often makes the task of proving a theorem easier and spares one the need to actually construct a mathematical object concretely. Nevertheless, anyone believing the validity of these tools accepts the existence of these objects without having to see them. It is the same with nondeterministic, nonreplicable phenomena. Finally, we would like to recall a famous verse from the Bible, John 20:29, where Jesus said "Blessed are they that have not seen, and yet have believed."

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References

[1] Ben Green, Terrence Tao, "The primes contain arbitrarily long arithmetic progressions," *arXiv:math/0404188*.
[2] See David Burton, *Elementary Number Theory* (New York: McGraw-Hill, 2005).
[3] Paul J. Cohen, "The Independence of the Continuum Hypothesis," *Proceedings of the National Academy of Sciences of the United States of America* 50 (6): 1143-1148.
[4] Paul J. Cohen, The Independence of the Continuum Hypothesis 2, *Proceedings of the National Academy of Sciences of the United States of America* 51 (1): 105-110.
[5] See Hebrews 11:6.
[6] Constance Reid, *Hilbert* (New York, Springer-Verlag 1996).